The Mathematics of the Channel Anamorphosis

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Abstract

An anamorphic amusement dating from the 16th and 17th century is described and discussed from a mathematical point of view. The Channel Anamorphosis displays two images which are cut into strips and arrayed on the alternate faces of long triangular prisms. The separate images are resolved when the array is viewed from one side or the other. Perspective corrections and finite-distance viewing of the images are discussed and analysed with examples. The 3-image channel anamorphosis is also briefly described.

1. Introduction. Anamorphic art began to appear soon after the invention of perspective and differs from it in that it requires a specific viewpoint so that the image returns back (‘ana’) to its normal form (‘morphe’). Holbein’s *Ambassadors* in the National Gallery in London [1] with the skull that appears when viewed from the right side is perhaps the most famous anamorphic painting. There are many types of anamorphic art, involving painting on surfaces like cones or requiring mirrors to restore the image [2]. The book *La Perspective Curieuse* by Jean-François Niceron, [3] although not often read carefully or in detail, has remained a central influence in the art of Anamorphosis as a source of practical methods for producing anamorphic art. Because of his early death, only the edition of 1638 is purely his work as is attested by the unity of style and graphics and the pervading mixture of didactic and speculative topics. On reading him with care one can only agree with Whitmore in describing him as “this most attractive man” [4].

Niceron constantly attempts to turn the anamorphosis to instructive, artistic and sacred uses and even, on occasion, for amusement. The latter is no more evident than in two “toys” which he describes in short sections of the book, one of which, the Channel Anamorphosis, is the subject of this paper.

The Channel Anamorphosis is well known and from beginnings as an expensive toy in the 16th century has remained with us as a cheap amusement or an advertising medium. Although it would seem to brook no mathematical elaboration there are, in fact, several interesting considerations that can be treated with simple mathematics.

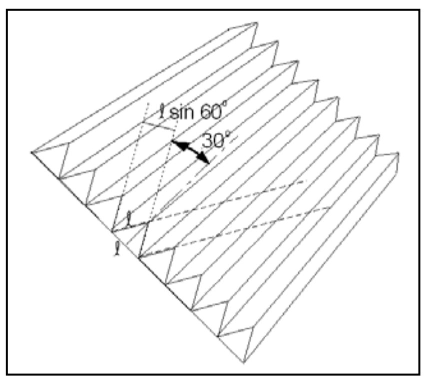

Figure 1: *The geometry of a 60/60/60 channel anamorphosis.*

2. Description of the Traditional Channel Anamorphosis. Among the many forms of anamorphosis, one of the easiest to construct and understand, at least in its simplest form, is the ‘Channel Anamorphosis” [1]. It consists of two images divided into a large number of vertical strips which are mounted on two of the three faces of long triangular prisms. “Image 1” is mounted on the

[1] There seems not to have been a consensus about nomenclature for this form. “Corrugated”, “ Stockade” , “Turning Pictures” and even (incorrectly) “lenticular” have been used; we prefer “Channel”.

faces tilted say, to the right, and “Image 2” on the intervening faces tilted left. The prisms are fastened to some substrate to form an array which is intended to be viewed from the right or left to display the images, but from straight on show only a confusion. The angles of the prism cross-section can be of any value but traditionally, from the earliest examples, they have been almost exclusively the 60°/60°/60° equilateral triangle. The array is shown in Fig. 1.

The uniform size and fixed tilt of the anamorphic faces imposes two restrictions on the viewing: 1. The images are to be viewed from infinity (or at least a distance which is large compared to the size of the array), and 2. there is only one correct angle for the line of sight to the array. These conditions arise from the requirement that all of the area of the strips must be visible for the observed image and no part of the other image is to be visible. The correct viewing angle for the equilateral case is at 30° on either side of the normal to the array as is shown in Fig. 1.

The device is remarkably effective at concealing the images in the frontal view, much more so than many of the other more elaborate types of anamorphosis. Because of their simplicity of construction, they are still encountered – chiefly in advertising on both a large and small scale.

Figure 2: *The channel portrait of Charles of Lorraine and his daughter by Ludovico Buti (1593). Courtesy of the Institute and Museum of the History of Science, Florence. Italy.*

The technique was well known in Shakespeare’s time as is demonstrated by Shickman [5]. An elegant example dating from the time of Shakespeare is shown in Fig.2. It is a portrait of Charles III, Duke of Lorraine and his daughter Christine who can be seen simultaneously by means of a plane mirror placed to observe the alternate image. The anamorphosis is on 60° prisms and was painted by Ludovico Buti in 1593.

3. Method of Construction. How the anamorphoses were prepared is described with some detail in books by both Niceron [3] and Vignola [6] who illustrate the technique with almost identical drawings. That of Niceron is reproduced as Fig. 3. Of course for inexpensive popular amusements the images can be drawn on paper, cut into strips and mounted either on wooden prisms of triangular cross-section or on card which has been folded into a corrugated array. Significantly Niceron discusses none of these methods; he seems to be addressing his instruction to those artisans who would be engaged in serious constructions such as the Buti painting in Fig. 2.

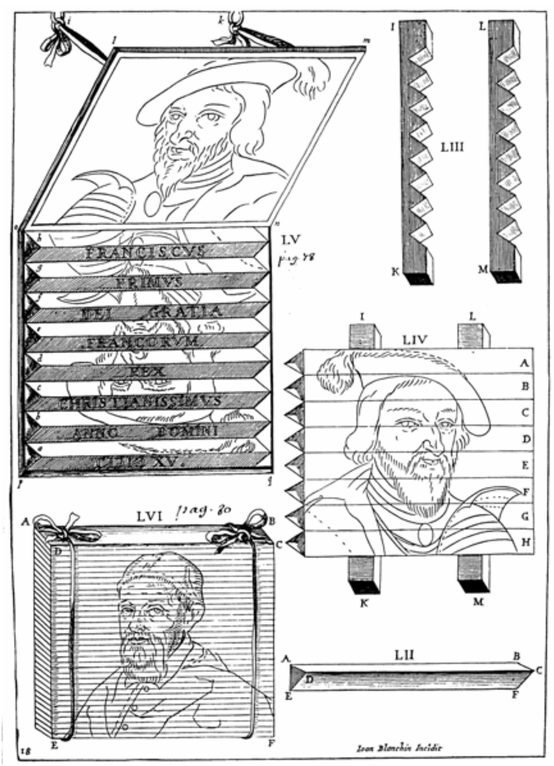

Figure 3: *Construction of Channel Anamorphoses as illustrated by Niceron.*

The instructions are to prepare a number of triangular prisms (See Fig. 3, LII) of isosceles cross-section with the face on which the picture will appear *slightly* smaller than the others. Niceron, who was a master of perspective, did not however depict something that was much different from a 60°/60°/60° equilateral triangle in ADE of Fig. 3/LII or the notches in Fig. 3/LIII. We take it then that neither he nor Vignola intended the cross-section to be anything like a 45°/45°/90° isosceles triangle.

This equilateral configuration has a particular advantage for the artisan. The prisms are mounted in the notched rails of Fig. 3/LIII to form a contiguous surface for painting as in Fig. 3/LIV. When it is finished then the second image can be painted on a fresh surface by rotating the prisms in their notched rails by 60°; that is not as convenient for any other combination of angles.

4. Perspective distortion due to the tilt of the strips. The choice of a 60° equilateral triangle for the prism cross-section introduces an anamorphic distortion which seems to have been recognized only rarely by the craftsmen of the 16th and 17th century. Referring to Fig. 1 it can be seen that each observed strip is seen inclined to the line of sight by 60° and so an image of length ℓ is foreshortened in the narrow dimension to $\ell \sin 60°$ = 0.866ℓ. This factor can be considered a constant across the width of an individual strip, and so can be compensated for by compressing the image in the long dimension by the same factor. Figure 4 is a photograph of a channel anamorphosis of circles; on the left is the unfolded anamorphosis. On the right is the view of both images (the upper by means of a mirror). The two columns on the right of the right-hand image, which appear as prolate ovals are, in fact, perfect circles in the anamorphosis, whereas the two columns on the left have been compressed in the horizontal direction by 0.866 and appear circular.

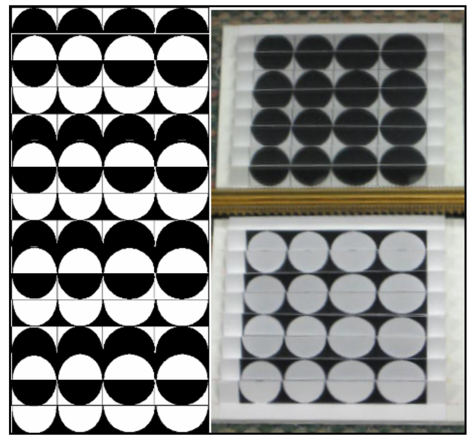

Figure 4: *Demonstration of the perspective distortion of the strips.*

This distortion can be completely eliminated by changing the form of the prisms from the 60° form to the cross-section of a 45°/45°/90° triangle with the image strips on the sides opposite the 45° angles, and the wide side opposite the 90°, in contact with the substrate. Now the viewing directions are at 45° with respect to the normal to the array, and the observed images are at right-angles to the line of sight, eliminating this perspective distortion entirely.

5. Perspective distortion due to the tilt of the array. The tilt of the array has a very large effect on the observed image which can be seen very clearly in Fig. 2. The portrait head of Charles is carried by a massive body distorted beyond reality. This distortion is partially caused by the tilt of the elements as discussed above but even more so by the tilt of the array.

Another well-known channel picture is that of Mary Queen of Scots and the skull shown in Fig. 5. In these images the ratio of the apparent vertical widths of the array (wide/narrow = near/far) is $d_2/d_1 = 1.19$. This means that the distance of the camera from the far end to that of the near end is in the same ratio. In addition, the centre of the frame (indicated by the intersection of the diagonals) is p = 47% of the image width from the most distant edge. Using these two parameters and considering only prism angles of 45° and 60° the

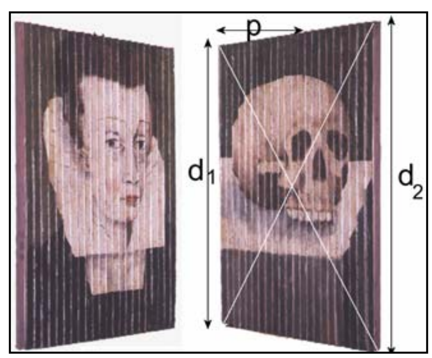

Figure 5: *Channel anamorphosis of Mary Queen of Scots and a skull.Courtesy of the National Portrait Gallery of Scotland, Edinburgh.*

value of k, the distance between the observation point and the array, can be determined. If the camera was aimed at the centre of the array, which seems natural, then k = 6 times the length of the frame at θ = 45° and 3.5 times for θ = 60° . These values are entirely reasonable; the distance is not too great which would be counter-productive for good viewing nor too close so as to exceed the acceptance cone of the camera's view. Using each of these values the perspective can be removed from the images and the face-on aspect ratio determined. For θ = 45° the ratio is 0.73 which very closely corresponds with the 33 cm by 24.8 cm dimensions of the back panel. For θ = 60° the ratio would be 0.60 which is impossible. This result would indicate that the cross-section of the prisms is 45°/45°/90° rather than 60°/60°/60°. [2]

Removing the perspective is equivalent to moving the picture to infinity and observing it normally from there. Since the angle of the prisms is 45° the sine correction as discussed in Section 4 is not required.

For this portrait to be presented accurately, it is not the portrait that should be put on the channel array but a plane anamorphosis of the portrait; this is discussed in the next section.

6. The "exact" solution for viewing from a finite distance. If the prisms are all the same size and have the same orientation then only viewing from infinity will produce an undistorted reconstruction, which we call the "inexact" solution. Since of necessity, the observation is from a finite distance then, when viewing the farther strips, they are seen to be partly occulted by the peaks of the adjacent and nearer prisms. Conversely, when viewing the nearer strips one is able to see a bit of the second image over the tops of the prisms. Corrections for this fault seem rarely to have been considered previously; however see the work of Bettini [7]. Although the solution is mathematically intractable, it can be solved numerically to give the "exact" construction. It requires a relaxation of both the rigid form for the prisms and their orientation. The solution for the case of "nearly-90°" prisms is illustrated in Fig. 6.

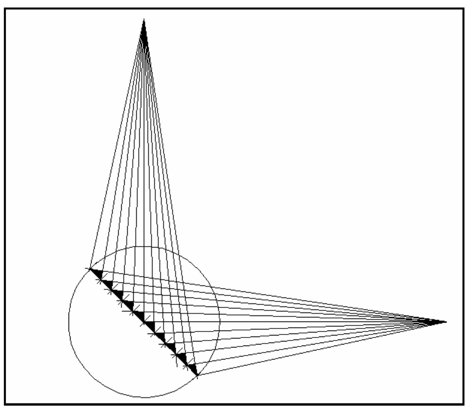

Figure 6: *The geometry of the "exact" construction for observation at 2× the array width.*

Since there is no formula for the form of the array then numerical methods are used. It is necessary to choose a model for the prisms and the array and then simply solve for the intersections of the sight-lines. Two models were considered:

A. The "equiangular" case where the individual prisms subtend equal angles at the observer's eye, and B. The "equally spaced" case where the substrate is divided into strips of equal width so that the width of a prism's footprint on the substrate is a constant and adjustments are made only to the angle itself.

Both models were investigated and gave equally good results with the case B being easier to calculate, so further consideration will be confined to it. The geometry of the specific case investigated is shown in Fig. 6. The observation distance from the centre of the array is taken to be 2× the length of the array and the observation directions to the same centre are fixed at 45° to the normal. It is only then a matter of writing the equations of the straight lines and solving for their intersection in pairs, to determine the points that define the dark triangles in Fig. 6. The exercise is easily carried out in a spreadsheet such

[2] Subsequent to writing this section we were informed by the Curator of the Scottish National Portrait Gallery, Ms. Nicola Kalinsky, that the prisms are indeed 90°.

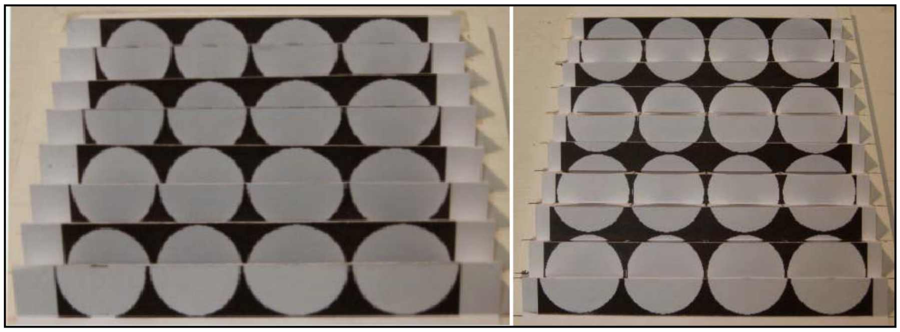

Figure 7: *Comparison of the inexact (left) and exact (right) anamorphoses.*

as EXCEL. As is expected, there is now an asymmetry introduced into the cross-section of the prisms as is most clearly seen in the outline for the prisms at each end of the array. In spite of these adjustments, the apex angle never deviates from 90° by more than 2% and so the transverse viewing of the channel images is almost unaltered, and the correction of Section 4 is not needed. Fig. 7 is a comparison of the inexact and exact models. On the left is the simple 45° channel anamorphosis properly intended for viewing from infinity but here photographed from a position from the centre at 2× its length. There is severe and increasing occultation of one portion of the image by the nearer portion. There are also horizontal mismatches but that is not the fault being addressed here. On the right in Fig. 7 is an "exact" version, and within the limits of construction, there is no exposure or overlap of the alternate image in the vertical direction.

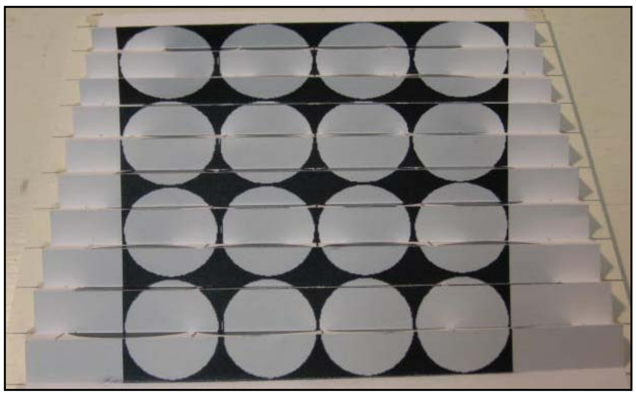

Figure 8: *The plane anamorphosis of the exact image in Fig. 7*

Of course what should be put on the channels of the anamorphosis is not the strip-elements of the picture, but strip-elements of a plane anamorphosis of the picture which compensates for the perspective due to the tilt of the array.[3] Accordingly an anamorphosis of the corrected picture was prepared using the proper equations (see Hunt et al [8]). The result of this final transformation is shown in Fig. 8. Now the mismatch in the horizontal direction has been eliminated and the array looks like a square array of circles viewed at right angles to the line of sight.

7. The Three-image Channel Anamorphosis. Channel anamorphoses can be constructed with three images but the added image imposes such severe constraints that the form is limited to being observed from infinity. Such pictures were common in the latter part of the 19^{th} century particularly for tourism mementos or with religious themes. They have sometimes been called "trisceneoramas".

The images are again arrayed in thin strips with Image-1 on the substrate itself observed in a direction normal to the array. The Images-2 and 3 are on alternating sides of thin walls between the elements of Image-1. It is vitally important that the height of the walls carrying Images-2 and 3 be exactly the same height as the distance between them. When viewing Image-1 the other two are not visible as the walls are being observed edge-on. On shifting to an angle of 45° Image-2 becomes visible

[3] Creating the rather unusual case of the anamorphosis of an anamorphosis!

on the sides of the walls facing the observer and the walls shield Image-1 from view. An identical situation exists for Image-3 when observing at 45° from the normal on the other side.

Since the images 2 and 3 on the vertical walls are tilted with respect to the direction of observation they suffer from the sine distortion discussed in Sec. 4. This time the angle of tilt is 45° so the compensating adjustment is to shrink all the strips of Images-2 and 3 in the long direction by sin 45° = 0.707. Since Image-1 is viewed straight-on it needs no adjustment. An example of this type of anamorphosis is shown in Fig. 9.

Figure 9: *A 3-image channel picture viewed from the left, centre and right*

Because of the particular geometry of the 3-image anamorphosis it is not possible to consider solutions for viewing at finite distance for example. Any adjustment to the height or angle of the walls for Image-2 will make correct observation of image 3 impossible and vice-versa.

There do not seem to be geometrical configurations that will permit the independent viewing of more than three static images.

Acknowledgements

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